For it then only need we show the simple theorem (Halsted's *Mensuration*, p. 16): When their sides tend indefinitely towards zero, the perimeter of the polygon inscribed increases, circumscribed decreases, toward the same limit.

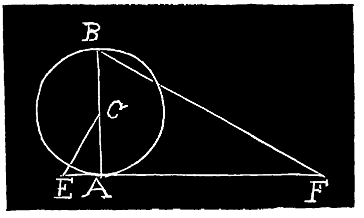
After that, we may deliberately and consciously declare by *definition* that this limit shall be what we will mean when we use the word *length* in connection with *circle*.

Here is no attempt to *prove* that the length of the circle is the limit of the lengths of perimeters of polygons. Any such attempt presupposes that we already know in some other way, or have in some other mathematical way defined what we are then already to mean by the length of the circle before we try to prove it equivalent to the limit for perimeters. What text book does this?

Can it be done? Let me try. (*Sect* is English for the German *strecke*).

Definition of length of an arc. We assume that with every arc is connected a sect such that if an arc be cut into two arcs, this sect is the sum of their sects; moreover this sect is not less than the chord of the arc, nor, if the arc be minor, is it greater than the sum of the sects on the tangents from the extremities of the arc to their intersection.

This sect we call *the length of the arc.*



Kochansky (1685) gave the following simple construction for the approximate length of the semicircle.

At the end-point A of the diameter BA, draw the tangent to the circle with center C. Take $ACE = \frac{1}{3}$ right angle. On the tangent, take $EF = 3AC$. Then BF is with great exactitude the length of the semicircle.

In fact, $BF = r[13\frac{1}{2} - 2(3)^{\frac{1}{2}}]^{\frac{1}{2}} = r3.1415$.

University of Texas, November, 1902.

AN ELEMENTARY ACCOUNT OF THE PICARD-VESSIOT THEORY.

By DR. SAUL EPSTEEN.

A theory of linear differential equations has been built up within the last twenty years which resembles the Galois theory of algebraic equations very closely.

In the Galois theory of equations we begin with an algebraic equation

$$(1) \qquad x^n + a_1 x^{n-1} + \dots + a_n = 0$$

and study the $n!$ permutations of the n roots $x_1, \dots, x_n$.

In the corresponding theory of differential equations we begin with the linear homogeneous differential equation

$$\text{(I)} \qquad \frac{d^n y}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+\dots+p_n y=0$$

and study the transformations of the continuous group

$$y'_i=\sum_{k=1}^{n} a_{ik}y_k \quad (i=1, \dots, n),$$

where $y_1, \dots, y_n$ form a fundamental system of integrals.

Many theorems analogous to well known theorems of algebra can be demonstrated in this new theory. For example, the theorem of Lagrange for indeterminate roots: "If a rational function $\phi(x_1, \dots, x_n)$ remains unaltered by all the substitutions which leave another rational function $\psi(x_1, \dots, x_n)$ unaltered, then ϕ can be expressed rationally in terms of ψ; $\phi=$Rat. $(\psi, a_1, \dots, a_n)$ has the following analog: If a rational function $\phi(y_1, \dots, y_n)$ of the integrals remains invariant under all the transformations which constitute the group of another rational function $\psi(y_1, \dots, y_n)$ then ϕ is rationally expressible in terms of ψ; $\phi=$Rat. $(\psi, p_1, \dots, p'_1, \dots, x)$."

As is well known, the group of an algebraic equation is defined by the two fundamental properties:

(*a*) Every rational function of the roots, which remains unaltered by all the substitutions of the group, is rationally known.

(*b*) Every rational function of the roots, which is rationally known, remains unaltered by all the substitutions of the group.

The parallel defining properties of the group of a differential equation are:

(*A*) Every rational function of the integrals (and their derivatives), which remains unaltered by all the transformations of the group, is rationally known.

(*B*) Every rational function of the integrals (and their derivatives), which is rationally known, remains unaltered by all the transformations of the group.

This brings us now to the question of formal and numerical invariance which is very much alike in both theories. The question will be illustrated by examples.

Let the algebraic equation be $x^4+1=0$ with the roots

$$x_1=\varepsilon,\ x_2=i\epsilon,\ x_3=-\varepsilon,\ x_4=-i\varepsilon \quad \left(\epsilon=\frac{1+i}{\sqrt{2}}\right).$$

The group of this equation is*

$$G=[\mathrm{I};\ (x_1x_2)(x_3x_4);\ (x_1x_3)(x_2x_4);\ (x_1x_4)(x_2x_3)].$$

*Bolza, *Bulletin American Mathematical Society*, First Series, Vol. 2, p. 99.

For the domain of rational numbers, the function x_1x_2 has the rational value -1. The substitutions of G which leave x_1x_2 formally invariant are $[I;\ (x_1x_2)(x_3x_4)]$, which form only a subgroup of the group of the equation. But all the substitutions of G leave x_1x_2 numerically invariant, since $x_1x_2=x_3x_4=-1$.*

Likewise in differential equations. The integral being regarded as functions of x, namely, $y_i=y_i(x)$ $(i=1, 2, \ldots, n)$, we consider, not the continuous group which leaves some function such as $\varphi(y_1, \ldots, y_n)=r(x)$ invariant as a function of the y's, but we consider a group which may change $\varphi(y_1, \ldots, y_n)$ to another function $\psi(y_1, \ldots, y_n)$ provided that *the numerical value is unaltered:* $\varphi=\psi=r(x)$. The value of x is not specified.

As an illustration, consider an irreducible linear homogeneous differential equation of the third order,

$$\frac{d^3y}{dx^3}+p_1\frac{d^2y}{dx^2}+p_2\frac{dy}{dx}+p_3y=0,$$

between a fundamental set of whose integrals there exists the *simple* rational relation

$$\varphi(y_1, y_2, y_3)\equiv y_2^2-y_1y_3=0.$$

If there were several such relations (as may well happen for equations of higher order) we must consider the group which leaves them all simultaneously invariant numerically.

The 3-parameter group

$$(2)\qquad \begin{aligned} y_1' &= a^2y_1+2aby_2+b^2y_3 \\ y_2' &= acy_1+(ad+bc)y_2+bdy_3 \\ y_3' &= c^2y_1+2cdy_2+d^3y_3 \end{aligned}$$

with $ad-bc=1$, leaves φ formally invariant since

$$y_2'^2-y_1'y_3'\equiv y_2^2-y_1y_3.$$

But it is not the group of the equation. The group of the equation (in the domain of the coefficients) is the 4-parameter group (2) where $ad-bc=$any constant different from zero. This latter group does not leave φ formally invariant; for now

$$y_2'^2-y_1'y_3'=(ad-bc)^3(y_2^2-y_1y_3).$$

*From this it must not be concluded that all the substitutions which leave x_1x_2 numerically unchanged: $G_8=[I;\ (x_1x_2);\ (x_3x_4);\ (x_1x_2)(x_3x_4);(x_1x_3)(x_2x_4);(x_1x_4)(x_2x_3);(x_1x_3x_2x_4);(x_1x_4x_2x_3)]$ form the group of the equation. The reason for this lies in (*b*); for we must consider not merely G_8 which leaves the *single* function $x_1x_4=-1$ numerically invariant, but we must consider a group (which turns out to be G) which leaves *every* function which has a rational value, unaltered. Thus $x_1x_2^2+x_3x_4^2=0$ has a rational value. Now the transposition (x_1x_2) of G_8 changes this to $x_2x_1^2+x_3x_4^2=\varepsilon(i-1)$, therefore G_8 is not the group of the equation. Notice that the group of the equation, G, will actually leave $x_1x_2^2+x_3x_4^2=0$ numerically unaltered.

The numerical value zero (for $\varphi=0$) is however unaltered by the group of the equation. (If φ were equal to $r(x)$ instead of to zero, the group of the equation would be (2) with $ad-bc=1$; for $ad-bc\neq1$, the numerical value would change from $r(x)$ to $(ad-bc)^3r(x)$.)

In the above example we assumed that the function $\varphi=y_2^2-y_1y_3$ was the invariant of the group. If the differential equation was of a higher order than the third the relation would be more complicated. Indeed, we might have a number of such relations involving not only the integrals but also their derivatives, say $\varphi_i(y_1\ldots.y_n,\ y_1'\ldots.y_n,\ y_1''\ldots.)$, $(i=1\ldots.p)$. The generalization can be carried still further. Notice that we assumed above that there exists algebraic relations between the integrals and their derivatives. Now this assumption is unnecessary, it may well happen that there are no such relations whatever, yet we continue still to speak of the group of the equation. An exposition of this question would be beyond the range set for this paper.

The group in question is technically called the *group of rationality* of the equation.

The literature of this subject is now quite extensive, but anyone familiar with the elements of the Galois theory will find it very easy to read the brief summary in Picard's *Traité de Analyse* III, last chapter, and to supplement it with Klein's *Höhere Geometrie* II, pp. 298-9.

The University of Chicago, October, 1902.

TWO SIMPLE CONSTRUCTIONS FOR FINDING THE FOCI OF AN HYPERBOLA, GIVEN THE ASYMPTOTES AND A POINT ON, OR A TANGENT TO, THE CURVE.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina.

The construction for a special position of the point is given first, as it is a linear construction.

Given the asymptotes of a hyperbola and the vertex A of the curve, to construct the foci.

The major axis is fixed, bisecting the angle between the asymptotes. Lay off, along an asymptote, from the center C of the hyperbola a distance $CD=CA=a$; draw a perpendicular to the asymptote at D. This meets the major axis (produced) in a focus F_1 (F_2).

For, the perpendicular F_1D from the focus $(ae,\ 0)$ upon the asymptote, whose equation is $y=\dfrac{b}{a}x$, is equal to b (by elementary principles). But $CF_1=\sqrt{(a^2+b^2)}$ and therefore $CD=a$.

The following style of argument (communicated to me in a letter a few